An Introduction to Hopf Algebra Gauge Theory

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(joint work with Catherine Meusburger)

Hopf Algebras in Kitaev's Quantum Double Models

Perimeter Institute

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Hopf algebra gauge theory

Goal: Conservative generalization of (lattice) gauge theory from groups to Hopf algebras.

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- Gauge theoretic understanding of existing models, e.g.:
  Turaev–Viro as regularization of 3d quantum gravity,
  Combinatorial quantization of Chern-Simons theory,
  Other gauge-theory-like models with specific algebras, bases, lattices, etc.

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- Kitaev models. (See Catherine Meusburger's talk, up next!)

Strategy

Take lattice gauge theory, and apply the monoidal functor

$$(\text{FinSet}, \times, 1) \to (\text{Vect}, \otimes, \mathbb{C})$$

to everything in sight.

FinSet	**Vect**
sets	vector spaces or better: *coalgebras*
groups	Hopf algebras
group actions	Hopf algebra modules

Then generalize to other fin. dim. Hopf algebras *(conservatively!)*

Result

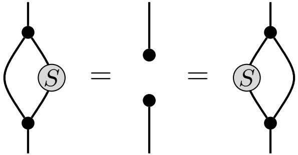
Reproduce Hamiltonian quantum Chern-Simons theory.

- topological invariant: **quantum moduli space**
  (analog of the moduli space of flat (classical) connections)
- derived "axiomatically" by generalizing gauge theory, rather than quantizing Poisson structures.

Hopf algebras

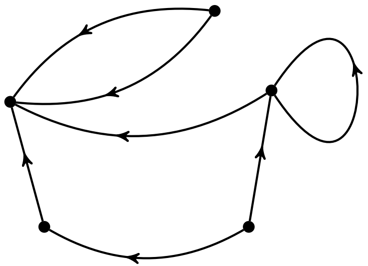
A **Hopf algebra** is a bialgebra H with **antipode** $S\colon H \to H$, drawn as:

Satisfying:



Lattice gauge theory

Graph with set V of vertices, set E of edges.



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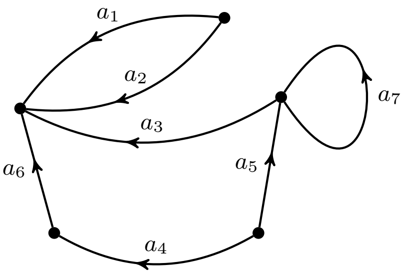

$a_1, \dots a_7 \in G$. G^E is the **set of connections**

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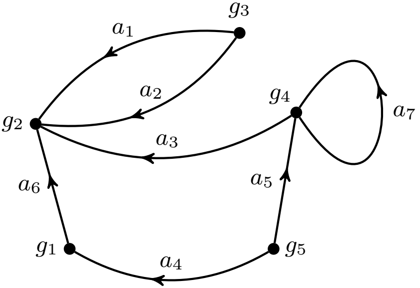

$a_1, \dots a_7 \in G$. G^E is the **set of connections**

$g_1, \dots g_5 \in G$. G^V is the **group of gauge transformations**.

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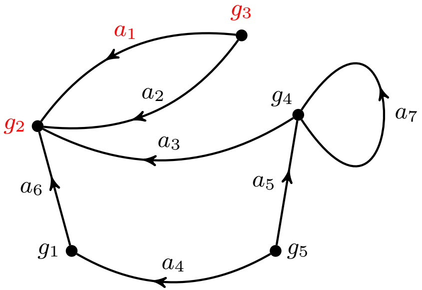

$a_1, \ldots a_7 \in G$. $\quad G^E$ is the **set of connections**
$g_1, \ldots g_5 \in G$. $\quad G^V$ is the **group of gauge transformations**.

Action of G^V on G^E: e.g. $a_1 \mapsto g_2 a_1 g_3^{-1}$.

Hopf algebra gauge theory from group gauge theory

Gauge theory for G

Gauge group G
Gauge trans.: $\mathcal{G} = G^V$
Connections: $\mathcal{A} = G^E$
Gauge action:

$$\triangleright \colon \mathcal{G} \times \mathcal{A} \to \mathcal{A}$$

Functions:

$$\mathcal{A}^* = \{f \colon \mathcal{A} \to \mathbb{C}\} \cong \mathbb{C}[G]^{*\otimes E}$$

Observables: $\mathcal{A}^*_{\text{inv}} \subset \mathcal{A}^*$ with

$$f(g \triangleright a) = f(a)$$

Gauge theory for $\mathbb{C}[G]$

Gauge Hopf algebra $\mathbb{C}[G]$
Gauge trans.: $\mathcal{G} = \mathbb{C}[G]^{\otimes V}$
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$$\triangleright \colon \mathcal{G} \otimes \mathcal{A} \to \mathcal{A}$$

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Goal: Generalize from $\mathbb{C}[G]$ to a finite-dimensional Hopf algebra H.

Gauge transformations

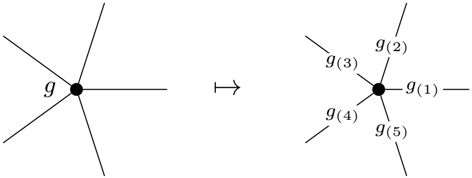
For a Hopf algebra H, the gauge action on connections should be

$$\triangleright \colon H^{\otimes V} \otimes H^{\otimes E} \to H^{\otimes E}$$

To make this *linear*, we need H's **comultiplication**:

$$\Delta \colon H \to H \otimes H$$

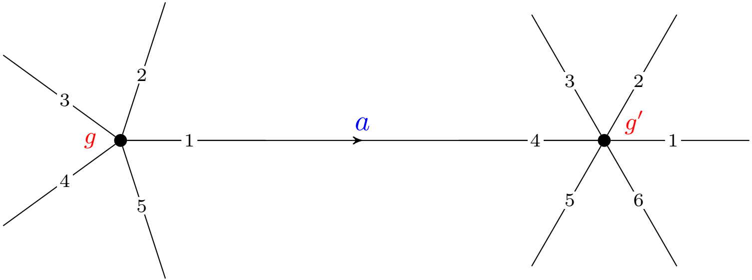
to "duplicate" vertex elements:



If H is not cocommutative, we need a total order at each vertex!

Gauge transformations

Otherwise, copy the gauge action as closely as possible:



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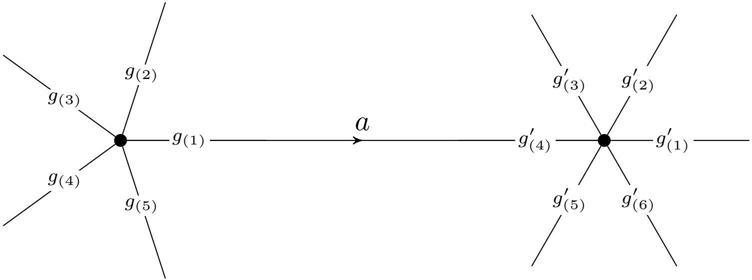

$$\cdots \otimes a \otimes \cdots \qquad \mapsto \qquad \cdots \otimes g'_{(4)} a S(g_{(1)}) \otimes \cdots$$

Ribbon Graphs

A graph with *cyclically ordered* edge-ends is a **ribbon graph**

$\implies$ surface with boundary.

$\implies$ closed surface, after sewing discs.

We've got a bit more... A graph with *totally ordered* edge-ends is a **ciliated ribbon graph**

End result is independent of 'ciliation' up to isomorphism, but the cyclic order matters

$\implies$ Hopf algebra gauge theory is fundamentally 2-dimensional.

So far, we've got...

Groups	**Hopf algebras**
Gauge group G	Gauge Hopf algebra H
Graph (V, E)	Ciliated ribbon graph (V, E)
Gauge trans.: $\mathcal{G} = G^V$	Gauge trans.: $\mathcal{G} = H^{\otimes V}$
Connections: $\mathcal{A} = G^E$	Connections: $\mathcal{A} = H^{\otimes E}$
Functions: $\mathcal{A}^* = \mathrm{Fun}(G) = \mathbb{C}[G]^*$	Functions: $\mathcal{A}^* = H^{*\otimes E}$

Observables

Groups

Observables are functions

$$f\colon G^E \to \mathbb{C}$$

that are **gauge invariant**:

$$f(g \triangleright a) = f(a) \qquad \forall g \in G^V$$

Functions form an *algebra* in an obvious way: $\mathcal{A}^* \cong \mathrm{Fun}(G)^{\otimes E}$

Observables form a *subalgebra*.

Hopf algebras

Observables are linear maps

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Observables are **not** a subalgebra, unless H is cocommutative!

Observables

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Observables are functions	**Observables** are linear maps
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$$f(g \triangleright a) = f(a) \qquad \forall g \in G^V$$	$$f(g \triangleright a) = \epsilon(g) f(a) \quad \forall g \in H^{\otimes V}$$
Functions form an *algebra* in an obvious way: $\mathcal{A}^* \cong \mathrm{Fun}(G)^{\otimes E}$	New approach: *generalize* algebra structure on $\mathcal{A}^* \cong H^{*\otimes E} \dots$
Observables form a *subalgebra*.	so that observables form a subalgebra.

But first, *why* doesn't the obvious algebra structure work?

Module coalgebras

To get a gauge-invariant *subalgebra* $\mathcal{A}^*_{\text{inv}} \subset \mathcal{A}^*$,
we need the action of $\mathcal{G}$ to preserve the algebra structure of $\mathcal{A}^*$
$\Leftrightarrow$ preserve the coalgebra structure of $\mathcal{A}$.

This means we need $\mathcal{A}$ to be a $\mathcal{G}$-**module coalgebra**:

$$\Delta(h \triangleright a) = \Delta(h) \triangleright \Delta(a) \qquad \epsilon(h \triangleright a) = \epsilon(h)\epsilon(a)$$

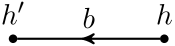
In the group case, this works automatically.

For Hopf algebras it does *NOT* work if we use the tensor product coalgebra structure on $\mathcal{A} \cong H^{\otimes E}$, unless H is cocommutative.

For example . . .

Example: Gauge theory "on the edge"

Graph with one edge, two vertices:



$$(h' \otimes h) \triangleright b = h' b S(h)$$

For a module coalgebra, we need:

$$\Delta((h' \otimes h) \triangleright a) \stackrel{?}{=} \Delta(h' \otimes h) \triangleright \Delta(a)$$

However, with the "obvious" coalgebra structure, we find

$$\begin{array}{ll} \text{LHS} & h'_{(1)} a_{(1)} S(h_{(2)}) \otimes h'_{(2)} a_{(2)} S(h_{(1)}) \\ \text{RHS} & h'_{(1)} a_{(1)} S(h_{(1)}) \otimes h'_{(2)} a_{(2)} S(h_{(2)}) \end{array}$$

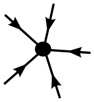
Fails because S is a coalgebra *anti*homomorphism:

$$\Delta(S(h)) = S(h)_{(1)} \otimes S(h)_{(2)} = S(h_{(2)}) \otimes S(h_{(1)})$$

But S is not the only problem ...

Example: Single vertex

Another example:



Edges all get acted on by gauge transformation at the vertex

But tensor product of module coalgebras is not generally a module coalgebra!

(More problems with ordering of factors in comultiplication...)

Quasitriangular Hopf Algebras

Need to relate Δ with Δ^{op}.

Suggests using a *quasitriangular* Hopf algebra, with R-matrix $R \in H \otimes H$.

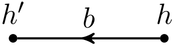
$$\Delta^{\text{op}}(h) = R\Delta(h)R^{-1} \qquad \forall h \in H$$

This helps. For example ...

Gauge theory “on the edge”

Gauge transformations: $\mathcal{G} = H \otimes H$ as a Hopf algebra.

Connections: $\mathcal{A} = H$ as a vector space.
$\mathcal{G}$-module structure:



$$(h' \otimes h) \triangleright b = h' b S(h)$$

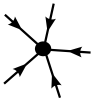
Coalgebra structure: (H, δ, ϵ)

$$\delta(a) = \Delta(a) R_{21}$$

This gives a *module coalgebra*.

Functions: Dual of $\mathcal{G}$-module coalgebra structure on $\mathcal{A}$
$\implies$ right $\mathcal{G}$-module algebra structure on $\mathcal{A}^*$
$\implies$ observables are a *subalgebra*.

Example: Single vertex



Solution is related to Majid's 'braided tensor products' of module (co)algebras ...

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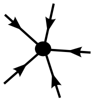

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But how do we figure this out systematically?

Plan: Axioms for Gauge Theory

Decide on axioms! What should a Hopf algebra gauge theory be like?

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Plan: Axioms for Gauge Theory

Decide on axioms! What should a Hopf algebra gauge theory be like?

- Mimic the gauge action from the group case
- Give a comodule algebra of connections
  $\implies$ module algebra of functions.
- Have an algebra of functions that is "local"

Local algebra

Since $\mathcal{A}^* \cong \bigotimes_E H^*$, we have embeddings for edges:

$$\iota_e \colon H^* \to \mathcal{A}^* \qquad\qquad \iota_e(\alpha) =: (\alpha)_e$$

and pairs of edges:

$$\iota_{ee'} \colon H^* \otimes H^* \to \mathcal{A}^* \qquad \iota_{ee'}(\alpha) =: (\alpha)_{ee'}$$

Say an algebra structure on $\mathcal{A}^* \cong H^{*\otimes E}$ with unit $1^{\otimes E}$ is **local** if:

(i) each $\iota_e(H^*)$ is a subalgebra of $\mathcal{A}^*$

(ii) each $\iota_{ee'}(H^* \otimes H^*)$ is a subalgebra of $\mathcal{A}^*$

(iii) If $e, e' \in E$ have no common vertex:
$(\alpha)_e \cdot (\beta)_{e'} = (\beta)_{e'} \cdot (\alpha)_e = (\alpha \otimes \beta)_{ee'}$ for all $\alpha, \beta \in H^*$

Hopf Algebra Gauge Theory

$\Gamma = (V, E)$ a ciliated ribbon graph, H a Hopf algebra.

Gauge theory on Γ with values in H consists of:

1 The Hopf algebra $\mathcal{G} = H^{\otimes V}$.

2 The vector space $\mathcal{A} = H^{\otimes E}$, equipped with a coalgebra structure such that the dual algebra structure on $\mathcal{A}^* \cong H^{*\otimes E}$ is *local*.

3 A left $\mathcal{G}$ module structure $\triangleright : \mathcal{G} \otimes \mathcal{A} \to \mathcal{A}$ on $\mathcal{A}$ such that:

(i) $\triangleright$ makes $\mathcal{A}$ into a $\mathcal{G}$ module coalgebra,

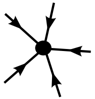
(ii) $\triangleright$ acts "as expected" for gauge transformations on single edges. That is: if $e \in E$ is not a loop, and $v \in V$ is not an endpoint of e:

$$(h)_v \triangleright (a)_e = \epsilon(h)(a)_e$$

$$(h)_{\mathbf{t}(e)} \triangleright (a)_e = (hk)_e$$

$$(h)_{\mathbf{s}(e)} \triangleright (a)_e = (aS(h))_e.$$

Example: Single vertex



(thought of as a degenerate 'graph')

For H quasi-triangular, there's an essentially unique algebra structure on $\mathcal{A}^*$ compatible with Hopf algebra gauge theory axioms:

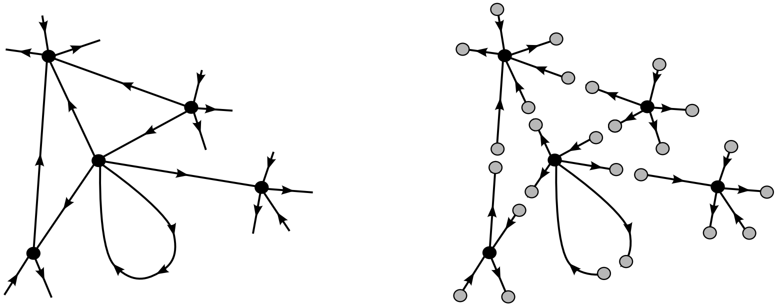
$$(\alpha)_i \cdot (\beta)_j = (\alpha \otimes \beta)_{ij} \qquad i < j$$
$$(\alpha)_i \cdot (\beta)_j = \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle \, (\alpha_{(2)} \otimes \beta_{(2)})_{ij} \qquad i > j.$$

and for $i = j$, we have two choices:

$$(\alpha)_i \cdot (\beta)_i = (\alpha\beta)_i \qquad \text{"normal"}$$
$$(\alpha)_i \cdot (\beta)_i = \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle \, (\beta_{(2)}\alpha_{(2)})_i \qquad \text{"twisted"}$$

independently for each edge end. (Reversing arrows requires *semisimple*, or more generally, *ribbon* Hopf alg.)

Strategy: Dissect the graph! (Locality lets us do this)



For each edge, one half-edge is “normal”, and the other is “twisted”. Comultiplication in H^* gives an injective linear map

$$G^* \colon \mathcal{A}^* \to \bigotimes_{v \in V} \mathcal{A}_v^*$$

Theorem: The image of G^* is a subalgebra and a $H^{\otimes V}$-submodule of $\bigotimes_v \mathcal{A}_v^*$. Pulling back this structure makes $\mathcal{A}^* := H^{*\otimes E}$ into the algebra of functions for a Hopf algebra gauge theory.

Results

- Hopf gauge theory determined by axioms: locality, module (co)algebra, and expected local gauge action.
- In any Hopf algebra gauge theory $\mathcal{A}^*_{inv} \subset \mathcal{A}^*$ is a subalgebra, the **algebra of observables**.
- Examples:
  - $\mathbb{C}[G] \implies$ Lattice gauge theory for G.
  - $D(H)$, single edge $\implies$ Heisenberg double of H
  - $D(H)$, single looped edge $\implies$ $D(H)$
- Algebra of functions coincides with the "lattice algebra" from combinatorial quantization of Chern-Simons theory. [Alekseev, Grosse, Schomerus 94], [Buffenoir, Roche 95]
- Topological invariant of the surface with boundary obtained from the ribbon graph.

Holonomy

If H is semisimple, then Hopf algebra gauge theory has a **holonomy functor**:

$$\mathrm{Hol}\colon \mathcal{P} \to \hom(H^{\otimes E}, H)$$

$\mathcal{P}$ is the **path groupoid** of the graph:

- objects: vertices
- morphisms: equivalence classes of edge-paths.

$\hom(H^{\otimes E}, H)$ is an **algebra** with multiplication

$$f \cdot g = m \circ (f \otimes g) \circ \Delta_{\otimes}$$

Associative algebra $\Leftrightarrow$ linear category with one object.

Curvature

- Holonomy around a face is **curvature**. A connection is flat if curvature at every face is 1.
- A Haar integral in H^* gives rise to a projector

$$P_{\text{flat}} \colon \mathcal{A}^*_{\text{inv}} \to \mathcal{A}^*_{\text{inv}}$$

  Image of P_{flat} is the **quantum moduli space** [Alekseev, Grosse, Schomerus '94], [Buffenoir Roche '95], [Meusburger, W]
- Topological invariant of the closed surface obtained from the ribbon graph. (Quantum analog of $\mathrm{hom}(\pi_1, G)/G$).

More information (and references)

C. Meusburger, D. K. Wise, Hopf algebra gauge theory on a ribbon graph, arXiv:1512.03966

Catherine's talk, after the coffee break.