An Introduction to Hopf Algebra Gauge Theory

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(joint work with Catherine Meusburger)

Hopf Algebras in Kitaev’s Quantum Double Models

Perimeter Institute

31 July – 4 August 2017
Goal: Conservative generalization of (lattice) gauge theory from groups to Hopf algebras.
Hopf algebra gauge theory

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**Why?**

- Deep ideas for groups deserve Hopf algebra analogues! (Hopf algebras *are* groups . . . in Vect.)
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- Gauge theoretic understanding of existing models, e.g.: Turaev–Viro as regularization of 3d quantum gravity, Combinatorial quantization of Chern-Simons theory, Other gauge-theory-like models with specific algebras, bases, lattices, etc.
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- Gauge theoretic understanding of existing models, e.g.: Turaev–Viro as regularization of 3d quantum gravity, Combinatorial quantization of Chern-Simons theory, Other gauge-theory-like models with specific algebras, bases, lattices, etc.
- Kitaev models. (See Catherine Meusburger’s talk, up next!)
Strategy

Take lattice gauge theory, and apply the monoidal functor

$$(\text{FinSet}, \times, 1) \to (\text{Vect}, \otimes, \mathbb{C})$$

to everything in sight.

<table>
<thead>
<tr>
<th>FinSet</th>
<th>Vect</th>
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<tbody>
<tr>
<td>sets</td>
<td>vector spaces</td>
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<tr>
<td>or better:</td>
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<td>coalgebras</td>
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<td>groups</td>
<td>Hopf algebras</td>
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<tr>
<td>group actions</td>
<td>Hopf algebra modules</td>
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Then generalize to other fin. dim. Hopf algebras (conservatively!)
Result

Reproduce Hamiltonian quantum Chern-Simons theory.

• topological invariant: quantum moduli space
  (analog of the moduli space of flat (classical) connections)

• derived “axiomatically” by generalizing gauge theory, rather than quantizing Poisson structures.
A **Hopf algebra** is a bialgebra $H$ with **antipode** $S: H \rightarrow H$, drawn as:

\[
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\]

Satisfying:
Graph with set $V$ of vertices, set $E$ of edges.
Lattice gauge theory

Graph with set \( V \) of vertices, set \( E \) of edges.

\[ a_1, \ldots, a_7 \in G. \quad G^E \text{ is the set of connections} \]
Lattice gauge theory

Graph with set $V$ of vertices, set $E$ of edges.

$a_1, \ldots, a_7 \in G$. $G^E$ is the set of connections
$g_1, \ldots, g_5 \in G$. $G^V$ is the group of gauge transformations.
Lattice gauge theory

Graph with set $V$ of vertices, set $E$ of edges.

Graph:

$a_1, \ldots a_7 \in G$.  $G^E$ is the set of connections
$g_1, \ldots g_5 \in G$.  $G^V$ is the group of gauge transformations.

Action of $G^V$ on $G^E$: e.g. $a_1 \mapsto g_2 a_1 g_3^{-1}$. 
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<td>Functions: $\mathcal{A}^* = { f : \mathcal{A} \rightarrow \mathbb{C} } \cong \mathbb{C}[G]^* \otimes E$</td>
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Goal: Generalize from $\mathbb{C}[G]$ to a finite-dimensional Hopf algebra $H$. 


## Hopf algebra gauge theory from group gauge theory

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### Functions:

$\mathcal{A}^* = \{ f: \mathcal{A} \to \mathbb{C} \} \cong \mathbb{C}[G]^* \otimes E$

### Observables:

$\mathcal{A}_{\text{inv}}^* \subset \mathcal{A}^*$ with

$f(g \triangleright a) = f(a)$

### Observables:

$\mathcal{A}_{\text{inv}}^* \subset \mathcal{A}^*$ with

$f(g \triangleright a) = \epsilon(g)f(a)$

Goal: Generalize from $\mathbb{C}[G]$ to a finite-dimensional Hopf algebra $H$. 
Gauge transformations

For a Hopf algebra $H$, the gauge action on connections should be

$$\triangleright : H \otimes V \otimes H \otimes E \to H \otimes E$$

To make this linear, we need $H$’s comultiplication:

$$\Delta : H \to H \otimes H$$

to “duplicate” vertex elements:

If $H$ is not cocommutative, we need a total order at each vertex!
Gauge transformations

Otherwise, copy the gauge action as closely as possible:
Gauge transformations

Otherwise, copy the gauge action as closely as possible:

\[ \cdots \otimes a \otimes \cdots \rightarrow \cdots \otimes g'_4 aS(g_1) \otimes \cdots \]
A graph with *cyclically ordered* edge-ends is a **ribbon graph**

$\longrightarrow$ surface with boundary.

$\longrightarrow$ closed surface, after sewing discs.

We’ve got a bit more... A graph with *totally ordered* edge-ends is a **ciliated ribbon graph**

End result is independent of ‘ciliation’ up to isomorphism, but the cyclic order matters

$\longrightarrow$ Hopf algebra gauge theory is fundamentally 2-dimensional.
Hopf gauge theory

So far, we’ve got...

**Groups**

- Gauge group $G$
- Graph $(V, E)$
- Gauge trans.: $\mathcal{G} = G^V$
- Connections: $\mathcal{A} = G^E$
- Functions: $\mathcal{A}^* = \text{Fun}(G) = \mathbb{C}[G]^*$

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Observables

Groups

Observables are functions

\[ f : G^E \to \mathbb{C} \]

that are \textbf{gauge invariant}:

\[ f(g \triangleright a) = f(a) \quad \forall g \in G^V \]

Functions form an \textit{algebra} in an obvious way: \( A^* \cong \text{Fun}(G)^E \)

Observables form a \textit{subalgebra}.

Hopf algebras

Observables are linear maps

\[ f : H^\otimes E \to \mathbb{C} \]

that are \textbf{gauge invariant}:

\[ f(g \triangleright a) = \epsilon(g)f(a) \quad \forall g \in H^{\otimes V} \]

Functions form an \textit{algebra} in an ‘obvious’ way: \( A^* \cong H^{*\otimes E} \)

Observables are \textbf{not} a subalgebra, unless \( H \) is cocommutative!
Observables

Groups

Observables are functions

\[ f : G^E \to \mathbb{C} \]

that are gauge invariant:

\[ f(g \triangleright a) = f(a) \quad \forall g \in G^V \]

Functions form an algebra in an obvious way: \( \mathcal{A}^* \cong \text{Fun}(G)^{\otimes E} \)

Observables form a subalgebra.

Hopf algebras

Observables are linear maps

\[ f : H^{\otimes E} \to \mathbb{C} \]

that are gauge invariant:

\[ f(g \triangleright a) = \epsilon(g)f(a) \quad \forall g \in H^{\otimes V} \]

New approach: generalize algebra structure on \( \mathcal{A}^* \cong H^{*\otimes E} \ldots \)

so that observables form a subalgebra.

But first, why doesn’t the obvious algebra structure work?
Module coalgebras

To get a gauge-invariant subalgebra $A_{\text{inv}}^* \subset A^*$, we need the action of $G$ to preserve the algebra structure of $A^*$ $\iff$ preserve the coalgebra structure of $A$.

This means we need $A$ to be a $G$-module coalgebra:

$$\Delta(h \triangleright a) = \Delta(h) \triangleright \Delta(a) \quad \epsilon(h \triangleright a) = \epsilon(h)\epsilon(a)$$

In the group case, this works automatically.

For Hopf algebras it does NOT work if we use the tensor product coalgebra structure on $A \cong H \otimes E$, unless $H$ is cocommutative.

For example ...
Example: Gauge theory “on the edge”

Graph with one edge, two vertices:

\[
\begin{array}{c}
h' \quad b \quad h \\
\bullet & \text{—} & \bullet
\end{array}
\]

\[(h' \otimes h) \triangleright b = h'bS(h)\]

For a module coalgebra, we need:

\[
\Delta((h' \otimes h) \triangleright a) \overset{?}{=} \Delta(h' \otimes h) \triangleright \Delta(a)
\]

However, with the “obvious” coalgebra structure, we find

- LHS: \( h'_{(1)} a_{(1)} S(h_{(2)}) \otimes h'_{(2)} a_{(2)} S(h_{(1)}) \)
- RHS: \( h'_{(1)} a_{(1)} S(h_{(1)}) \otimes h'_{(2)} a_{(2)} S(h_{(2)}) \)

Fails because \( S \) is a coalgebra antihomomorphism:

\[
\Delta(S(h)) = S(h)_{(1)} \otimes S(h)_{(2)} = S(h_{(2)}) \otimes S(h_{(1)})
\]

But \( S \) is not the only problem ...
Example: Single vertex

Another example:

Edges all get acted on by gauge transformation at the vertex

But tensor product of module coalgebras is not generally a module coalgebra!

(More problems with ordering of factors in comultiplication...)
Need to relate $\Delta$ with $\Delta^{\text{op}}$.

Suggests using a quasitriangular Hopf algebra, with $R$-matrix $R \in H \otimes H$.

$$\Delta^{\text{op}}(h) = R\Delta(h)R^{-1} \quad \forall h \in H$$

This helps. For example ...
Gauge theory “on the edge”

Gauge transformations: $\mathcal{G} = H \otimes H$ as a Hopf algebra.

Connections: $\mathcal{A} = H$ as a vector space.
$\mathcal{G}$-module structure:

\[
\begin{align*}
  h' & \quad b \quad h \\
\end{align*}
\]

\[(h' \otimes h) \triangleright b = h'bS(h)\]

Coalgebra structure: $(H, \delta, \epsilon)$

\[
\delta(a) = \Delta(a) R_{21}
\]

This gives a module coalgebra.

Functions: Dual of $\mathcal{G}$-module coalgebra structure on $\mathcal{A}$

$\implies$ right $\mathcal{G}$-module algebra structure on $\mathcal{A}^*$

$\implies$ observables are a subalgebra.
Example: Single vertex

Solution is related to Majid’s ‘braided tensor products’ of module (co)algebras …
Example: Single vertex

Solution is related to Majid’s ‘braided tensor products’ of module (co)algebras . . .

But how do we figure this out systematically?
Plan: Axioms for Gauge Theory

Decide on axioms! What should a Hopf algebra gauge theory be like?
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- Mimic the gauge action from the group case
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- Mimic the gauge action from the group case
- Give a comodule algebra of connections
  \[ \Rightarrow \] module algebra of functions.
Decide on axioms! What should a Hopf algebra gauge theory be like?

• Mimic the gauge action from the group case
• Give a comodule algebra of connections
  $\implies$ module algebra of functions.
• Have an algebra of functions that is “local”
Local algebra

Since $A^* \cong \bigotimes_E H^*$, we have embeddings for edges:

$$\iota_e : H^* \to A^* \quad \iota_e(\alpha) =: (\alpha)_e$$

and pairs of edges:

$$\iota_{ee'} : H^* \otimes H^* \to A^* \quad \iota_{ee'}(\alpha) =: (\alpha)_{ee'}$$

Say an algebra structure on $A^* \cong H^* \otimes E$ with unit $1 \otimes E$ is local if:

(i) each $\iota_e(H^*)$ is a subalgebra of $A^*$

(ii) each $\iota_{ee'}(H^* \otimes H^*)$ is a subalgebra of $A^*$

(iii) If $e, e' \in E$ have no common vertex:

$$\quad (\alpha)_e \cdot (\beta)_{e'} = (\beta)_{e'} \cdot (\alpha)_e = (\alpha \otimes \beta)_{ee'}$$

for all $\alpha, \beta \in H^*$
Hopf Algebra Gauge Theory

Γ = (V, E) a ciliated ribbon graph, H a Hopf algebra.

Gauge theory on Γ with values in H consists of:
1. The Hopf algebra \( \mathcal{G} = H \otimes V \).
2. The vector space \( \mathcal{A} = H \otimes E \), equipped with a coalgebra structure such that the dual algebra structure on \( \mathcal{A}^* \cong H^* \otimes E \) is local.
3. A left \( \mathcal{G} \) module structure \( \triangleright : \mathcal{G} \otimes \mathcal{A} \to \mathcal{A} \) on \( \mathcal{A} \) such that:
   (i) \( \triangleright \) makes \( \mathcal{A} \) into a \( \mathcal{G} \) module coalgebra,
   (ii) \( \triangleright \) acts “as expected” for gauge transformations on single edges. That is: if \( e \in E \) is not a loop, and \( v \in V \) is not an endpoint of \( e \):

\[
(h)_v \triangleright (a)_e = \epsilon(h)(a)_e \\
(h)_{t(e)} \triangleright (a)_e = (hk)_e \\
(h)_{s(e)} \triangleright (a)_e = (aS(h))_e.
\]
Example: Single vertex

(though of as a degenerate ‘graph’)

For $H$ quasi-triangular, there’s an essentially unique algebra structure on $A^*$ compatible with Hopf algebra gauge theory axioms:

$$(\alpha)_i \cdot (\beta)_j = (\alpha \otimes \beta)_{ij} \quad i < j$$

$$(\alpha)_i \cdot (\beta)_j = \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle (\alpha_{(2)} \otimes \beta_{(2)})_{ij} \quad i > j.$$ 

and for $i = j$, we have two choices:

$$(\alpha)_i \cdot (\beta)_i = (\alpha \beta)_i \quad \text{“normal”}$$

$$(\alpha)_i \cdot (\beta)_i = \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle (\beta_{(2)} \alpha_{(2)})_i \quad \text{“twisted”}$$

independently for each edge end. (Reversing arrows requires *semisimple*, or more generally, *ribbon* Hopf alg.)
Strategy: Dissect the graph! (Locality lets us do this)

For each edge, one half-edge is “normal”, and the other is “twisted”. Comultiplication in \( H^* \) gives an injective linear map

\[
G^* : \mathcal{A}^* \rightarrow \bigotimes_{v \in V} \mathcal{A}_v^*
\]

**Theorem:** The image of \( G^* \) is a subalgebra and a \( H^* \otimes V \)-submodule of \( \bigotimes_v \mathcal{A}_v^* \). Pulling back this structure makes \( \mathcal{A}^* := H^* \otimes E \) into the algebra of functions for a Hopf algebra gauge theory.
Results

• Hopf gauge theory determined by axioms: locality, module (co)algebra, and expected local gauge action.

• In any Hopf algebra gauge theory $A^*_{inv} \subset A^*$ is a subalgebra, the algebra of observables.

• Examples:
  • $\mathbb{C}[G] \implies$ Lattice gauge theory for $G$.
  • $D(H)$, single edge $\implies$ Heisenberg double of $H$
  • $D(H)$, single looped edge $\implies$ $D(H)$

• Algebra of functions coincides with the “lattice algebra” from combinatorial quantization of Chern-Simons theory. [Alekseev, Grosse, Schomerus 94], [Buffenoir, Roche 95]

• Topological invariant of the surface with boundary obtained from the ribbon graph.
If $H$ is semisimple, then Hopf algebra gauge theory has a **holonomy functor**:

$$\text{Hol}: \mathcal{P} \rightarrow \text{hom}(H \otimes E, H)$$

$\mathcal{P}$ is the **path groupoid** of the graph:

- objects: vertices
- morphisms: equivalence classes of edge-paths.

$\text{hom}(H \otimes E, H)$ is an **algebra** with multiplication

$$f \cdot g = m \circ (f \otimes g) \circ \Delta \otimes$$

Associative algebra $\Leftrightarrow$ linear category with one object.
Curvature

- Holonomy around a face is **curvature**. A connection is flat if curvature at every face is 1.
- A Haar integral in $H^*$ gives rise to a projector

$$P_{\text{flat}} : \mathcal{A}_{\text{inv}}^* \to \mathcal{A}_{\text{inv}}^*$$

Image of $P_{\text{flat}}$ is the **quantum moduli space**
[Alekseev, Grosse, Schomerus ’94], [Buffenoir Roche ’95], [Meusburger, W]

- Topological invariant of the closed surface obtained from the ribbon graph. (Quantum analog of hom($\pi_1$, $G$)/$G$).

Catherine’s talk, after the coffee break.